TAMU Math Circle, Spring 2012
Domino puzzles, invariants, and walking along the grids and bridges
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Main ideas discussed in March 3 meeting

1. Domino puzzles

A domino is just a 1×2 (or 2×1) rectangle. We are going to investigate here questions of the following type: given a gridded shape can we cover it by an appropriate number of dominoes (assuming that our dominoes exactly matches the size of the two adjacent squares on our grid?) To cover means that every square of the grid is covered exactly by one domino and the dominoes do not go out of the given gridded shape.

1.1 Getting started

1. Can we cover a 5×5 grid by dominoes?

Answer: No! If the gridded shape can be covered by dominoes then the number of the squares in the grid must be even. In other words, the condition that the number of the squares in the grid is even is a necessary condition for the existence of a covering of this shape by dominoes.

Now we consider only gridded shapes with even number of squares. Can we cover any of this shape by dominoes?

2. Can we cover by dominoes a 3×3 grid with a corner square removed?

Answer Of course, yes, see the picture:
3. Can we cover by dominoes a $3 \times 3$ grid with a middle square in the bottom row removed?

[Image]

**Answer** No!

One way: Getting the contradiction by placing dominoes by necessity. $3 \times 3$ grid is too small so two bottom corners can be covered by dominoes only in one way. Then the central square must be covered like in the picture:

[Image]

And the remained two upper corners obviously cannot be covered by dominoes without overlapping. So we cannot cover our shape although it has even number of squares.

1.2 There is a MAGIC HERE

4. Can we cover by dominoes a $7 \times 7$ grid with a square in the bottom row removed as in the next figure?

[Image]
5. Can we cover by dominoes a $8 \times 8$ grid with two diametrically opposite corners removed?

Answer for both problem is No! In this situation we have much more room to place dominoes, so if we will try to get contradiction by placing dominoes by necessity, we will quickly get lost because too many cases have to be considered.

There is much clever, in fact, MAGICAL IDEA, THE CHESSBOARD COLORING:

Let us color our grid as a chessboard. For example, let us start with the grid of Problem 3.

Look at the picture: we have 5 black (=maroon in the picture) squares and only 3 white squares! Any domino piece covers exactly one black square and one white square. Therefore,

If the grid can be covered by dominoes then, after making a chessboard coloring, the number of black square in the grid must be equal to the number of the white squares in the grid,

$$
\# \text{black squares} = \# \text{white squares}
$$

(1)

Hence equation (1) gives another necessary condition for the existence of a covering of a gridded shape by dominoes.

Remark 1. Note that this necessary condition is stronger than the previous necessary condition (that the number of squares in the grid must be even): if (1) holds then the total number of squares is even.
Let us apply our magic necessary condition (1) for Problems 3-5:

1. In Problem 3 \#black squares = 5 and \#white squares = 3. \(5 \neq 3\) \(\Rightarrow\) a domino covering is impossible.

2. In Problem 4: Color the whole \(7 \times 7\) grid as a chessboard such that one corner is black (\(\Rightarrow\) all other corners are black). Then in the whole \(7 \times 7\) grid \#black squares = 25 and \#white squares = 24. If we remove a white square (as in Problem 4) then in the resulting grid \#black squares = 25 and \#white squares = 23 \(\Rightarrow\) the domino covering is impossible.

3. In Problem 5: Color the whole \(8 \times 8\) grid as a chessboard. In the whole \(8 \times 8\) grid \#black squares = \#white squares = 32. Two diametrically opposite corners have the same color. Hence, if we remove them the magic condition (1) does not hold \(\Rightarrow\) the domino covering is impossible.

Remark 2. As the matter of fact, we proved much more than was asked in Problems 4 and 5:

Generalization of Problem 4: Given a \(n \times m\) grid with odd \(mn\) (or, equivalently, such that both \(m\) and \(n\) are odd), if it is colored by a chessboard coloring such that one (and therefore any corner) is black, then the gridded shape obtained by removing one white square from this grid cannot be covered by dominoes.

Generalization of Problem 5: Given a \(n \times m\) grid with even \(mn\) (or, equivalently, such that either \(m\) or \(n\) is even), the gridded shape obtained by removing two squares of the same color from this grid cannot be covered by dominoes.

1.3 Cutting a snake

6. Consider the standard \(8 \times 8\) chessboard. Remove one black square and one white square from the chessboard (for example as in the next figure). Can we cover the remaining board with 31 dominoes?
Answer: Yes!

We are going to demonstrate an algorithm how to built such covering.

Remark 3. In general, the word “algorithm” means the list of rules for solving certain class of problems (implementing certain class of tasks). This word comes from the name of a persian scientist Al-Khwarizmi (∼ 780 – ∼ 850 A.D.) who was the founder of Algebra.

Draw a “closed snake” (a “warm”, a “necklace”) passing through all squares of the $8 \times 8$ grid ones as in the picture:

![Snake Diagram](image)

The chessboard coloring colors the snake. Removing two non-subsequent (along the snake) squares we cut the snake on two pieces (as if one removes two points from a circle then we will end up with two connected pieces). If the removed squares are of different colors, then for each of this two (colored) pieces of the snake the first and the last squares have different colors. Therefore each piece can be covered by dominoes placing them one-after-one along the piece (you can rectify each piece, i.e. make it straight, then it consists of even number of squares, then place dominoes one-after one along this “straight piece” then return to the original shape by urning some dominoes). Hence the whole shape can be colored by dominoes.

If we remove two subsequent (along the snake) squares we get only one piece only after cutting with the first and the last squares having different colors. Then again we can cover this piece (and therefore our shape) by dominoes placing them one-after another along this piece.

Remark 4. The same conclusion can be made for $m \times n$ grid with even $mn$. Also similar statement is valid for $m \times n$ grid with odd $mn$, namely, if we color by a chessboard coloring such that one (and therefore any corner) is black, then the gridded shape obtained by removing from one black square from this grid can be...
covered by dominoes. To prove it we use the similar argument with a non-closed snake as in the picture:

![Diagram of dominoes](image)

**Remark 5.** We proved that if our shape is $8 \times 8$ grid with two squares removed then the condition that $\#\text{black squares} = \#\text{white squares}$ is not only necessary but sufficient for the existence of a domino covering.

**Question** Is this condition sufficient if we remove some number of black squares and then the same number of white squares?

**Answer:** No. Counterexample: remove all squares except of two non-diagonal corners (in this case they are of different colors).

Think what other natural questions can we ask at this point?

2. **Transformation of tables and invariants**

2.1 **Formulation**

9 numbers are arranged in the squares of a $3 \times 3$ table as follows

a) \[
\begin{array}{ccc}
 5 & 8 & 3 \\
 4 & -2 & -3 \\
 6 & 10 & 7 \\
\end{array}
\]

b) \[
\begin{array}{ccc}
 7 & 4 & -3 \\
 -5 & 2 & 6 \\
 -1 & 8 & 10 \\
\end{array}
\]

You are allowed to add or subtract the same number from any two squares in a table having a common edge. For example, by adding 2 to the numbers in the first two squares of the second column of the table of item (a), you obtain the new table:

\[
\begin{array}{ccc}
 5 & 8 & 3 \\
 4 & -2 & -3 \\
 6 & 10 & 7 \\
\end{array} \quad \rightarrow \quad \begin{array}{ccc}
 5 & 10 & 3 \\
 4 & 0 & -3 \\
 6 & 10 & 7 \\
\end{array}
\]
7. For each of the tables (a) and (b) answer the following question: Is it possible after applying such transformations several times to obtained the $3 \times 3$ table with zeros in every square (the zero table), namely:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\]

If yes, show how? If not, show why?

**Answer for table a)** is YES!
The algorithm of doing this is the *step-by-step elimination*: step-by-step killing (i.e. making zero) the numbers in the table along the pass shown in the following figure by applying the permitted transformations:

\[
\begin{array}{c}
\begin{array}{|ccc|}
\hline
& & \\
\hline
& & \\
\hline
\end{array}
\end{array}
\]

Then

\[
\begin{array}{ccc}
5 & 8 & 3 \\
4 & -2 & -3 \\
6 & 10 & 7
\end{array}
\rightarrow
\begin{array}{ccc}
0 & 3 & 3 \\
4 & -2 & -3 \\
6 & 10 & 7
\end{array}
\rightarrow
\begin{array}{ccc}
0 & 0 & 0 \\
4 & -2 & -3 \\
6 & 10 & 7
\end{array}
\rightarrow
\begin{array}{ccc}
0 & 0 & 0 \\
4 & 1 & 0 \\
6 & 10 & 7
\end{array}
\rightarrow
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\rightarrow
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\rightarrow
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\rightarrow
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 2
\end{array}
\]

**Answer for table a)** is NO!
If we will apply the procedure of step-by-step elimination as in item a) we will get at the end the following table
Although we did not get the zero table THIS IS NOT A PROOF of impossibility to transform the table in b) to the zero table. It only shows that it cannot be done by the procedure of step-by-step elimination but maybe there is another way of doing it.

How to prove nonexistence rigirously? OUR OLD FRIEND-chessboard coloring.

2.2 A new idea: The notion of invariant, a quantity that is preserved by a permitted transformation

The solution is based on the following two observations:

(a) When we make a permitted transformation the two squares involved in it (i.e. those in which the numbers are changed) are of different colors.

(b) If you add the same number to two numbers then the difference between them is preserved: \((a + c) - (b + c) = a - b\) (the difference of ages of two brothers is preserved).

To any table of numbers assign the following quantity:

\[ I = \text{sum of numbers in black squares} - \text{sum of numbers in white squares}. \] (2)

By the two reasons given above the quantity \(I\) DOES NOT CHANGE after applying a permitted transformation. A quantity that is preserved under any permitted transformation is called an invariant. So in our problem the quantity \(I\) defined by (2) is an invariant.

If a table can be transformed to the zero table by a sequence of permitted transformations then \(I\) of this table must be equal to 0 (because \(I\) is an invariant and for the zero matrix \(I = 0\)). In other words the condition \(I = 0\) is a necessary condition for a possibility to transform the table to the zero table by a sequence of permitted transformations.

For the table of item b) \(I = 2\), therefore the table in item b) cannot be transformed to the zero table by a sequence of permitted transformations

2.3 Necessary versus sufficient

Question: Is the condition \(I = 0\) sufficient?

Answer: Yes! If we apply the method of the successive elimination for a table with invariant \(I\) then the resulting table will be
So, if $I = 0$, we arrived to the zero matrix.

**Question** Given any two $3 \times 3$ tables what is a necessary and sufficient condition for a possibility to transform one table to another by a sequence of permitted transformation.

**Answer:** $I$ of the first table = $I$ of the second table (justify).

### 3. Walking on the grids

Consider a $3 \times 3$ grid. You want to start in the top left square and without raising your pencil pass through all 9 squares once each. The figure below shows one way to do it. You must pass through side, not through corners.

8. Find a path passing through all the squares starting in the middle square of the grid, or prove no such path exist;

**Answer** We can do it as in the picture:

9. Find a path passing through all the squares starting in the middle square of the bottom row, or prove no such path exist.

**Answer** We cannot do it. We guess this after several experiments but how to prove our answer: our old friend, the chessboard coloring.

To any path we can assign a word consisting of two letters $B$ and $W$ depending on the color of the squares that we passing when we move along the path. For example, the word $BWBWBWBW$ corresponds to a path in the figure of the problem 8. The point is that in any such word a letter $B$ (if it is not the last letter) must be followed by $W$ and vice versa. So, if we start with the $W$ (i.e. on a white square)
and the number of squares is odd (as in the problem) then we must end up with W (i.e. on a white square), which means that \# of white squares = \# black squares + 1. However, in our situation \# of white squares = 3 and \# of white squares = 5 \imp Contradiction.

This is basically what we discussed on March 3. And we will continue on March 24 to try to solve several exiting problems on invariants, coloring (more general than the chessboard coloring) and graphs i.e. sets of points some of which are connected by lines, like islands connected by bridges, cities connected by roads, phones connected by wires etc.